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Eigenvalues of Casimir operators for $gl(m/\infty)$

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A full set of Casimir operators for the Lie superalgebra $gl(m/\infty)$ is constructed and shown to be well defined in the category O_{FS} generated by the highest weight irreducible representations with only a finite number of non-zero weight components. The eigenvalues of these Casimir operators are determined explicitly in terms of the highest weight. Characteristic identities satisfied by certain (infinite) matrices with entries from $gl(m/\infty)$ are also determined.

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I. INTRODUCTION

During the last years the infinite dimensional Lie algebras and Lie superalgebras play an important role in several areas of theoretical and mathematical physics[1−9]. They have applications in the theory of integrable field equations, string theory, two-dimensional statistical models. In addition these algebras are of interest as examples of Kac-Moody Lie (super-)algebras of infinite type.

However, for these algebras such a fundamental concept as Casimir invariants has not yet been determined. The present paper is a step in solving this problem.

We construct a full set of Casimir operators for the infinite dimensional general linear Lie superalgebra $gl(m/\infty)$ corresponding to the natural matrix realization, namely

$$gl(m/\infty) = \{X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in M_{m\times m}, B \in M_{m\times\infty}, C \in M_{\infty\times m}, D \in M_{\infty\times\infty},$$
$$all\ but\ a\ finite\ number\ of\ X_{ij} \in \mathbf{C}\ are\ zero\}, \tag{1}$$

where $M_{p\times q}$ is the space of all $p \times q$ complex matrices. The even subalgebra $gl(m/\infty)_{\bar{0}}$ has $B = 0$ and $C = 0$; the odd subspace $gl(m/\infty)_{\bar{1}}$ has $A = 0$ and $D = 0$.

A basis for the Lie superalgebra $gl(m/\infty)$ is given by the Weyl generators $E_{ij},\ i,j = -m+1, -m+2, \ldots, 0, 1, \ldots$. Assign to each index i a degree $\langle i \rangle$, which is zero for $i \in -\mathbf{Z}_+$ and 1 for $i \in \mathbf{N}$ (see the notation at the end of the Introduction). Then the generator E_{ij} is even (resp. odd), if $\langle i \rangle + \langle j \rangle$ is an even (resp. odd) number. The multiplication ($\equiv$the supercommutator) $[\![\ ,\ ]\!]$ of $gl(m/\infty)$ is given by the linear extension of the relations:

$$[\![E_{ij}, E_{kl}]\!] = \delta_{jk}E_{il} - (-1)^{(\langle i\rangle+\langle j\rangle)(\langle k\rangle+\langle l\rangle)}\delta_{il}E_{kj}. \tag{2}$$

We will consider the category O_{FS} generated by all highest weight irreducible $gl(m/\infty)$ modules $V(\Lambda)$ with a finite number of non-zero highest weight components Λ_i of the highest weight

$$\Lambda \equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k, 0, 0, \ldots) \equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k, \dot{0}). \tag{3}$$

The highest weight Λ of $V(\Lambda)$ uniquely characterized the module and satisfies the conditions:

$$\Lambda_i - \Lambda_{i+1} \in \mathbf{Z}_+, \quad \forall i \neq 0. \tag{4}$$

Denote by H the Cartan subalgebra of $gl(m/\infty)$. The dual space H^* of H is described by the forms $\varepsilon_i,\ i = -m+1, -m+1, \ldots,$ where $\varepsilon_i : X \to A_{ii}$, for $-m+1 \leq i \leq 0$ and $\varepsilon_i : X \to D_{ii},\ \ \forall i \in \mathbf{N}$, and X is given by (1) only for diagonal X. On H^* there is a bilinear form (,) defined by

$$\begin{aligned} &(\epsilon_i, \epsilon_j) = \delta_{ij}, && for\ \ -m+1 \leq i,j \leq 0; \\ &(\epsilon_i, \epsilon_j) = 0, && for\ \ -m+1 \leq i \leq 0\ \ and\ \ j \in \mathbf{N}; \\ &(\epsilon_i, \epsilon_j) = -\delta_{ij}, && for\ \ \ i,j \in \mathbf{N}. \end{aligned} \tag{5}$$

The roots $\varepsilon_i - \varepsilon_j$ $(i \neq j)$ of $gl(m/\infty)$ are the non-zero weights of the adjoint representation. The positive roots are those given by the set:

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j | i < j,\ i,j = -m+1, -m+2, \ldots\}. \tag{6}$$

Define

$$\rho = \frac{1}{2} \sum_{i=-m+1}^{0} (1-2i-2m)\epsilon_i + \frac{1}{2}\sum_{i=1}^{\infty}(1-2i+2m)\epsilon_i. \tag{7}$$

Let D_n be the set of $gl(m/\infty)$ weights:

$$D_n = \{\nu | \nu = (\nu_{-m+1}, \ldots, \nu_0, \nu_1, \ldots, \nu_n, \dot{0}),\ \ \nu_i \in \mathbf{Z}_+,\ \ i = -m+1, -m+2, \ldots, n-1,\ \ \nu_n \in \mathbf{N}\}, \tag{8}$$

and let $D_n^+ \subset D_n$ be the subset of integral dominant weights in D_n :

$$D_n^+ = \{\nu | \nu \in D_n,\ \nu_i - \nu_{i+1} \in \mathbf{Z}_+,\ \ \forall i \neq 0\}. \tag{9}$$

Note that if ν is a weight in $V(\Lambda),\ \Lambda \in D_k^+$, then $\nu \in D_n$, for some $n \in \mathbf{Z}_+$.

In Section II we construct a full set of Casimir operators convergent on each module $V(\Lambda)$. The eigenvalues of these Casimir invariants for all modules from the category O_{FS} are computed in Section III. In Section IV we present a derivation of the polynomial identities satisfied by certain matrices with entries from $gl(m/\infty)$.

Throughout the paper we use the following notation:

irrep(s) - irreducible representation(s);

$\mathbf{C}$ - the complex numbers;

$\mathbf{Z}_+$ - all non-negative integers;

$\mathbf{N}$ - all positive integers;

$U(A)$ - the universal enveloping algebra of A;

$\langle i \rangle = \begin{cases} 0 & \text{for } i \in -\mathbf{Z}_+ \\ 1 & \text{for } i \in \mathbf{N}. \end{cases}$

II. CONSTRUCTION OF CASIMIR OPERATORS

An obvious invariant for $gl(m/\infty)$ is the first order invariant

$$I_1 = \sum_{i=-m+1}^{\infty} E_{ii}. \tag{10}$$

It is not clear, however, how to construct appropriate higher order Casimir operators for $gl(m/\infty)$. Let us first consider the second order invariant $I_2^{(m,n)}$ of $gl(m/n)$:

$$I_2^{(m,n)} = \sum_{i,j=-m+1}^{n} (-1)^{\langle j \rangle} E_{ij}E_{ji} = \sum_{i,j=-m+1}^{0} E_{ij}E_{ji} - \sum_{i,j=1}^{n} E_{ij}E_{ji} + \sum_{i=1}^{n}\ \sum_{j=-m+1}^{0} E_{ij}E_{ji}$$

$$
\begin{aligned}
&-\sum_{i=-m+1}^{0}\sum_{j=1}^{n}E_{ij}E_{ji}=\sum_{i=-m+1}^{0}\sum_{j<i=-m+1}^{0}E_{ij}E_{ji}+\sum_{i=-m+1}^{0}\sum_{j>i=-m+1}^{0}E_{ij}E_{ji}+\sum_{i=-m+1}^{0}E_{ii}^2\\
&-\sum_{i=1}^{n}\sum_{j<i=1}^{n}E_{ij}E_{ji}-\sum_{i=1}^{n}\sum_{j>i=1}^{n}E_{ij}E_{ji}-\sum_{i=1}^{n}E_{ii}^2+2\sum_{i=1}^{n}\sum_{j=-m+1}^{0}E_{ij}E_{ji}-\sum_{i=-m+1}^{0}\sum_{j=1}^{n}(E_{ii}+E_{jj})\\
&=2\sum_{i=-m+1}^{0}\sum_{j<i=-m+1}^{0}E_{ij}E_{ji}+\sum_{i=-m+1}^{0}\sum_{j>i=-m+1}^{0}(E_{ii}-E_{jj})+\sum_{i=-m+1}^{0}E_{ii}^2-2\sum_{i=1}^{n}\sum_{j<i=1}^{n}E_{ij}E_{ji}\\
&-\sum_{i=1}^{n}\sum_{j>i=1}^{n}(E_{ii}-E_{jj})-\sum_{i=1}^{n}E_{ii}^2+2\sum_{i=1}^{n}\sum_{j=-m+1}^{0}E_{ij}E_{ji}-n\sum_{i=-m+1}^{0}E_{ii}-m\sum_{i=1}^{n}E_{ii}\\
&=2\sum_{i=-m+1}^{n}\sum_{j<i=-m+1}^{n}(-1)^{\langle j\rangle}E_{ij}E_{ji}+\sum_{i=-m+1}^{0}E_{ii}(E_{ii}+1-m-2i)-\sum_{i=1}^{n}E_{ii}(E_{ii}+1+n-2i)\\
&-n\sum_{i=-m+1}^{0}E_{ii}-m\sum_{i=1}^{n}E_{ii}\\
&=2\sum_{i=-m+1}^{n}\sum_{j<i=-m+1}^{n}(-1)^{\langle j\rangle}E_{ij}E_{ji}+\sum_{i=-m+1}^{n}(-1)^{\langle i\rangle}E_{ii}(E_{ii}+1-2i)-(m+n)I_1^{(m,n)}\\
&=2\sum_{i=-m+1}^{n}\sum_{j<i=-m+1}^{n}(-1)^{\langle j\rangle}E_{ij}E_{ji}+\sum_{i=-m+1}^{n}(-1)^{\langle i\rangle}E_{ii}(E_{ii}+1-2i)-2mI_1^{(m,n)}+(m-n)I_1^{(m,n)},
\end{aligned}
\tag{11}
$$

where $I_1^{(m,n)}\equiv\sum_{i=-m+1}^{n}E_{ii}$ is the first order invariant of $gl(m/n)$. Due to the last term in (11) the $gl(m/n)$ second order invariant diverges as $n\to\infty$. Eliminating the last term in (11) (the rest of the expression is also an invariant) and taking the limit $n\to\infty$ one obtains the following quadratic Casimir for $gl(m/\infty)$:

$$
I_2=2\sum_{i=-m+1}^{\infty}\sum_{j<i=-m+1}^{\infty}(-1)^{\langle j\rangle}E_{ij}E_{ji}+\sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}E_{ii}(E_{ii}+1-2i)-2mI_1, \tag{12}
$$

which is convergent (see formula (21)) on the category O_{FS} of irreps considered. On $V(\Lambda),\ \Lambda\in D_k^+,\ \ I_2$ takes constant value

$$
\chi_\Lambda(I_2)=\sum_{i=-m+1}^{k}\Big((-1)^{\langle i\rangle}\Lambda_i(\Lambda_i+1-2i)-2m\Lambda_i\Big)=(\Lambda,\Lambda+2\rho). \tag{13}
$$

This consideration shows how to construct the higher order Casimir operators of $gl(m/\infty)$.

Introduce to this end the characteristic matrix

$$
A_i^{\ j}=(-1)^{\langle i\rangle\langle j\rangle}E_{ji}. \tag{14}
$$

Define the powers of the matrix A recursively by

$$
(A^q)_i^{\ j}=\sum_{k=-m+1}^{\infty}A_i^{\ k}(A^{q-1})_k^{\ j},\qquad [(A^0)_i^{\ j}\equiv\delta_{ij}]. \tag{15}
$$

Using induction and the $gl(m/\infty)$ commutation relations (2) one obtains:

Proposition 1:

$$[\![E_{kl},(A^q)_i^{\ j}]\!] = (-1)^{(\langle k\rangle+\langle l\rangle)\langle i\rangle}\left(\delta_{lj}(A^q)_i^{\ k} - \delta_{ik}(A^q)_l^{\ j}\right). \tag{16}$$

[]

Therefore the matrix supertraces

$$str(A^q) \equiv \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}(A^q)_i^{\ i} \tag{17}$$

are formally Casimir operators. They are, however, divergent except for $q=1$ in which case we obtain the first order invariant (10). Our purpose is to construct a full set of Casimir invariants which are well defined and convergent on the category O_{FS}.

Theorem 1: *The Casimir operators defined recursively by*

$$\begin{aligned} I_1 &= \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}A_i^{\ i} = str(A);\\ I_q &= \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}\left[(A^q)_i^{\ i} - I_{q-1}\right] = str\left[A^q - I_{q-1}\right] \end{aligned} \tag{18}$$

form a full set of convergent $gl(m/\infty)$ Casimir operators on each module $V(\Lambda)\in O_{FS}$. []

Observe that the operators I_q are indeed Casimir invariants (see *Proposition 1*). Then it remains to prove they are convergent on the category O_{FS}. We will do this by induction. Consider first the case $q=2$:

$$\begin{aligned}
I_2 &\equiv \sum_{j=-m+1}^{\infty}(-1)^{\langle j\rangle}\left[(A^2)_j^{\ j} - I_1\right] = \sum_{j=-m+1}^{0}\left[\sum_{i=-m+1}^{\infty}E_{ij}E_{ji} - I_1\right]\\
&- \sum_{j=1}^{\infty}\left[\sum_{i=-m+1}^{\infty}E_{ij}E_{ji} - I_1\right] = \sum_{j=-m+1}^{0}\ \sum_{i=-m+1}^{0}E_{ij}E_{ji} + \sum_{j=-m+1}^{0}\ \sum_{i=1}^{\infty}E_{ij}E_{ji} - mI_1\\
&- \sum_{j=1}^{\infty}\ \sum_{i=-m+1}^{0}E_{ij}E_{ji} - \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}E_{ij}E_{ji} - I_1\right]\\
&= 2\sum_{i=-m+1}^{0}\ \sum_{j<i=-m+1}^{0}E_{ij}E_{ji} + \sum_{i=-m+1}^{0}E_{ii}(E_{ii}+1-m-2i) - mI_1 + 2\sum_{j=-m+1}^{0}\ \sum_{i=1}^{\infty}E_{ij}E_{ji}\\
&- \sum_{j=1}^{\infty}\ \sum_{i=-m+1}^{0}(E_{ii}+E_{jj}) - \sum_{j=1}^{\infty}\left[2\sum_{i>j=1}^{\infty}E_{ij}E_{ji} + \sum_{i<j=1}^{\infty}(E_{ii}-E_{jj}) + E_{jj}^2 - I_1\right]\\
&= 2\sum_{i=-m+1}^{\infty}\ \sum_{j<i=-m+1}^{\infty}(-1)^{\langle j\rangle}E_{ij}E_{ji} + \sum_{i=-m+1}^{0}E_{ii}(E_{ii}+1-m-2i) - mI_1 - m\sum_{i=1}^{\infty}E_{ii}\\
&- \sum_{j=1}^{\infty}\left[\sum_{i<j=1}^{\infty}(E_{ii}-E_{jj}) + E_{jj}^2 - \sum_{i=1}^{\infty}E_{ii}\right]
\end{aligned}$$

$$= 2 \sum_{i=-m+1}^{\infty} \sum_{j<i=-m+1}^{\infty} (-1)^{\langle j \rangle} E_{ij} E_{ji} + \sum_{i=-m+1}^{0} E_{ii}(E_{ii} + 1 - 2i) - 2mI_1 - \sum_{i=1}^{\infty} E_{ii}(E_{ii} + 1 - 2i), \tag{19}$$

which agrees with the definition (12).

Now let $v \in V(\Lambda),\ \Lambda \in D_k^+,$ be an arbitrary weight vector. Then the weight of v has the form

$$\nu = (\nu_{-m+1}, \nu_{-m+2}, \ldots, \nu_0, \ldots, \nu_r, \dot{0}). \tag{20}$$

Since

$$A_i^{\,j} v = (-1)^{\langle i \rangle \langle j \rangle} E_{ji} v = 0, \quad \forall i > r, \tag{21}$$

the second order invariant I_2 is convergent on each $V(\Lambda) \in O_{FS}$ [c.f. formula (13)].

Applying *Proposition 1* and (21), for $i > r$ one obtains

$$\begin{aligned}
(A^q)_i^{\,i} v &= \sum_{j=-m+1}^{\infty} A_i^{\,j} (A^{q-1})_j^{\,i} v = \sum_{j=-m+1}^{\infty} (-1)^{\langle i \rangle \langle j \rangle} E_{ji} (A^{q-1})_j^{\,i} v \\
&= \sum_{j=-m+1}^{\infty} (-1)^{\langle i \rangle \langle j \rangle} \left\{ (-1)^{(\langle j \rangle + \langle i \rangle)\langle j \rangle} \left[(A^{q-1})_j^{\,j} - (A^{q-1})_i^{\,i} \right] v + (-1)^{(\langle i \rangle + \langle j \rangle)} (A^{q-1})_j^{\,i} E_{ji} v \right\} \\
&= \sum_{j=-m+1}^{\infty} (-1)^{\langle j \rangle} \left[(A^{q-1})_j^{\,j} - (A^{q-1})_i^{\,i} \right] v.
\end{aligned} \tag{22}$$

For the case $q = 2$ we have

$$(A^2)_i^{\,i} v = \sum_{j=-m+1}^{\infty} (-1)^{\langle j \rangle} \left[A_j^{\,j} - A_i^{\,i} \right] v = \sum_{j=-m+1}^{\infty} E_{jj} v = I_1 v, \ \ \forall i > r \tag{23}$$

so that

$$\left((A^2)_i^{\,i} - I_1 \right) v = 0, \ \forall i > r, \tag{24}$$

which is another proof for the convergence of I_2. More generally

Proposition 2: For any weight vector $v \in V(\Lambda)$, and $q \in \mathbf{N}$ there exist $r \in \mathbf{N}$ such that

$$\left((A^q)_i^{\,i} - I_{q-1} \right) v = 0, \ \ \forall i > r. \tag{25}$$

Proof: We proceed by induction. Assume v has weight ν as in (20). Formula (25) is valid for $q = 2$ (24). Let the result be true for a given q, i.e.

$$(A^q)_i^{\,i} v = I_{q-1} v, \ \ \forall i > r.$$

Then (see (22))

$$(A^{q+1})_i^{\,i} v = \sum_{j=-m+1}^{\infty} (-1)^{\langle j \rangle} \left[(A^q)_j^{\,j} - (A^q)_i^{\,i} \right] v = \sum_{j=-m+1}^{\infty} (-1)^{\langle j \rangle} \left[(A^q)_j^{\,j} - I_{q-1} \right] v = I_q v, \quad \forall i > r, \tag{26}$$

which proves (25). []

I_q (18) is convergent on each $V(\Lambda)$ for $q=2$. Assume it is well defined and convergent on $V(\Lambda)$ for a given q. Then, with v as in (25), we have

$$I_{q+1}v \equiv \sum_{i=-m+1}^{\infty} (-1)^{\langle i\rangle} \left[(A^{q+1})_i^{\ i} - I_q\right] v = \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle} \left[(A^{q+1})_i^{\ i} - I_q\right] v$$

$$= \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle} (A^{q+1})_i^{\ i} v + (r-m) I_q v. \tag{27}$$

Therefore I_{q+1} is convergent and well defined on $V(\Lambda)$.

This completes the (inductive) proof of *Theorem 1.*

III. EIGENVALUE FORMULA FOR CASIMIR OPERATORS

In this section we apply our previous results to evaluate the spectrum of the operators (18).

Let $v \in V(\Lambda)$, be an arbitrary vector of weight $\nu = (\nu_{-m+1}, \nu_{-m+2} \ldots, \nu_0, \nu_1, \ldots, \nu_r, \dot{0})$. Then, keeping in mind *Proposition 1,* the fact that $(A^{q-1})_k^{\ j}$ has weight $\varepsilon_j - \varepsilon_k$ under the adjoint representation of $gl(m/\infty)$ and that all vectors of $V(\Lambda)$ have weight components ν_i in $\mathbf{Z}_+$, we must have for $j \le r$

$$(A^{q-1})_k^{\ j} v = 0, \quad \forall k > r. \tag{28}$$

Therefore

$$(A^q)_i^{\ j} v = \sum_{k=-m+1}^{\infty} A_i^{\ k} (A^{q-1})_k^{\ j} v = \sum_{k=-m+1}^{r} A_i^{\ k} (A^{q-1})_k^{\ j} v. \tag{29}$$

Proceeding recursively we may therefore write

$$(A^q)_i^{\ j} v = (\bar{A}^q)_i^{\ j} v, \quad \forall i,j = -m+1, -m+2, \ldots, r, \tag{30}$$

where $(\bar{A})_i^{\ j} = (-1)^{\langle i\rangle\langle j\rangle} E_{ji}, \ \ \forall i,j = -m+1, \ldots, r$, is the $gl(m/r)$ characteristic matrix, and the powers of the matrix $\bar{A}$ are defined by (15) with $i,j,k = -m+1, \ldots, r$ and $\bar{A}$ instead of A. It follows then that the formula (27) can be written as:

$$I_q v = \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle} \left[(\bar{A}^q)_i^{\ i} - I_{q-1}\right] v = \left[I_q^{(m,r)} - (m-r) I_{q-1}\right] v, \tag{31}$$

with

$$I_q^{(m,r)} = \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle} (\bar{A}^q)_i^{\ i}, \tag{32}$$

being the q^{th} order invariant of $gl(m/r)$. Formula (31) is valid $\forall q \in \mathbf{N}$, which gives a recursion relation for the I_q with initial condition

$$I_1 v = \chi_\Lambda(I_1) v. \tag{33}$$

In particular it follows from (31) that the invariants I_q are certainly convergent on all weight vectors $v \in V(\Lambda)$.

To determine the eigenvalues of I_q let $v = v_\Lambda^+$ be the highest weight vector of the $V(\Lambda)$ module and let

$$\Lambda = (\bar{\Lambda}, \dot{0}) \in D_k^+, \quad \bar{\Lambda} \equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k). \tag{34}$$

Then for the eigenvalues of the I_q one obtains the recursion relation (see (31)):

$$\chi_\Lambda(I_q) = \chi_{\bar{\Lambda}}({I_q}^{(m,k)}) - (m-k)\chi_\Lambda(I_{q-1}), \quad \chi_\Lambda(I_1) = \sum_{i=-m+1}^{k} \Lambda_i, \tag{35}$$

where $\chi_{\bar{\Lambda}}({I_q}^{(m,k)})$ is the eigenvalue of the q^{th} order invariant (32) of $gl(m/k)$ on the irreducible $gl(m/k)$ module with highest weight $\bar{\Lambda}$; the latter is given explicitly by[10]

$$\chi_{\bar{\Lambda}}({I_q}^{(m,k)}) = \sum_{i=-m+1}^{k} (-1)^{\langle i \rangle} \alpha_i^q \prod_{j \neq i=-m+1}^{k} \left(\frac{\alpha_i - \alpha_j + (-1)^{\langle j \rangle}}{\alpha_i - \alpha_j} \right), \tag{36}$$

where

$$\alpha_i = (-1)^{\langle i \rangle} (\Lambda_i - i + 1) - m.$$

Therefore we obtain for the eigenvalues of the Casimir operators I_q

$$\chi_\Lambda(I_q) = \sum_{i=-m+1}^{k} (-1)^{\langle i \rangle} P_q(\alpha_i) \prod_{j \neq i=-m+1}^{k} \left(\frac{\alpha_i - \alpha_j + (-1)^{\langle j \rangle}}{\alpha_i - \alpha_j} \right), \tag{37}$$

for suitable polynomials $P_q(x)$ which, from Eq. (35), satisfy the recursion relation

$$P_q(x) = x^q - (m-k)P_{q-1}(x), \quad P_1(x) = x. \tag{38}$$

In particular

$$P_2(x) = x^2 - (m-k)x = x\frac{x^2 - (m-k)^2}{x + (m-k)}; \tag{39a}$$

$$P_3(x) = x^3 - (m-k)(x^2 - (m-k)x) = x\frac{x^3 + (m-k)^3}{x + (m-k)}, \tag{39b}$$

and more generally, it is easily established by induction that

$$P_q(x) = x\frac{x^q - (-1)^q (m-k)^q}{x + (m-k)}. \tag{40}$$

Thus we have

Theorem 2: *The eigenvalues of the Casimir operators I_q (18), on the irreducible $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D_k^+$ are given by*

$$\chi_\Lambda(I_q) = \sum_{i=-m+1}^{k} (-1)^{\langle i \rangle} \alpha_i \left(\frac{\alpha_i^q - (-1)^q (m-k)^q}{\alpha_i + (m-k)} \right) \prod_{j \neq i=-m+1}^{k} \left(\frac{\alpha_i - \alpha_j + (-1)^{\langle j \rangle}}{\alpha_i - \alpha_j} \right),$$

$$\textit{where } \ \alpha_i = (-1)^{\langle i \rangle} (\Lambda_i - i + 1) - m. \tag{41}$$

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IV. POLYNOMIAL IDENTITIES

Let Δ be the comultiplication on the enveloping algebra $U[gl(m/\infty)]$ of $gl(m/\infty)$ ($\Delta(E_{ij}) = E_{ij} \otimes 1 + 1 \otimes E_{ij},\;\; i,j = -m+1, -m+2, \ldots$ with 1 being the unit in $U[gl(m/\infty)]$). Applying Δ to the second order Casimir operator (12) of $gl(m/\infty)$ we obtain:

$$\Delta(I_2) = I_2 \otimes 1 + 1 \otimes I_2 + 2 \sum_{i,j=-m+1}^{\infty} (-1)^{\langle j \rangle} E_{ij} \otimes E_{ji}. \tag{42}$$

Therefore

$$\sum_{i,j=-m+1}^{\infty} (-1)^{\langle j \rangle} E_{ij} \otimes E_{ji} = \frac{1}{2} \left[\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2 \right]. \tag{43}$$

Denote by $\pi_{\varepsilon_{-m+1}}$ the irrep of $gl(m/\infty)$ afforded by $V(\varepsilon_{-m+1})$. The weight spectrum for the vector module $V(\varepsilon_{-m+1})$ consists of all weights $\varepsilon_i,\;\; i = -m+1, -m+2, \ldots,$ each occurring exactly once. Denote by $e_{ij},\; i,j = -m+1, -m+2, \ldots$ the generators on this space

$$\pi_{\varepsilon_{-m+1}}(E_{ij}) = e_{ij}, \tag{44}$$

with e_{ij} an elementary matrix.

Introduce the characteristic matrix

$$A = \frac{1}{2} (\pi_{\varepsilon_{-m+1}} \otimes 1) \left[\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2 \right]. \tag{45}$$

Therefore

$$A_k{}^l = \sum_{i,j=-m+1}^{\infty} (-1)^{(\langle i \rangle + \langle j \rangle)\langle l \rangle} \pi_{\varepsilon_{-m+1}}(E_{ij})_{kl} (-1)^{\langle j \rangle} E_{ji} = (-1)^{\langle k \rangle \langle l \rangle} E_{lk}. \tag{46}$$

The matrix A is the infinite matrix introduced in Sec. II (see (14)) and the entries of the matrix powers A^q are given recursively by (15). We will see that the characteristic matrix satisfies a polynomial identity acting on the $gl(m/\infty)$ module $V(\Lambda),\;\Lambda \in D_k^+$. Let π_Λ be the representation afforded by $V(\Lambda)$. From Eq. (45), acting on $V(\Lambda)$ we may interpret A as an invariant operator on the tensor product module $V(\varepsilon_{-m+1}) \otimes V(\Lambda)$:

$$A \equiv \frac{1}{2} (\pi_{\varepsilon_{-m+1}} \otimes \pi_\Lambda) \left[\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2 \right]. \tag{47}$$

Following Ref. 11 it is easy to see that the tensor product space admits a filtration of submodules

$$V(\varepsilon_{-m+1}) \otimes V(\Lambda) = V_{k+1} \supseteq V_k \supseteq \ldots V_0 \supseteq \ldots \supseteq V_{-m+1} \supseteq (0), \tag{48}$$

where each factor module $M_i = V_i/V_{i+1}$, if non-zero, is indecomposable and cyclically generated by a highest weight vector of weight $\Lambda + \varepsilon_i$. We emphasize that M_i is only non-zero when $\Lambda + \varepsilon_i$ is integral dominant. Then it follows that the generalized eigenvalues of A on the tensor product space are given by

$$\begin{aligned} \frac{1}{2} \left[\chi_{\Lambda+\varepsilon_i}(I_2) - \chi_{\varepsilon_{-m+1}}(I_2) - \chi_\Lambda(I_2) \right] &= \frac{1}{2} \left[(\Lambda + \varepsilon_i, \Lambda + \varepsilon_i + 2\rho) - (\varepsilon_{-m+1}, \varepsilon_{-m+1} + 2\rho) - (\Lambda, \Lambda + 2\rho) \right] \\ &= (-1)^{\langle i \rangle} (\Lambda_i + 1 - i) - m, \end{aligned} \tag{49}$$

(see *Theorem 2*). Thus we have

Theorem 3: *On each $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D_k^+$ the characteristic matrix satisfies the polynomial identity*

$$\prod_{i=-m+1}^{k+1} (A - \alpha_i) = 0, \tag{50}$$

with $\alpha_i = (-1)^{\langle i \rangle} (\Lambda_i + 1 - i) - m$ *the characteristic roots.* []

Note that the characteristic identities (50) are the $gl(m/\infty)$ counterpart of the polynomial identities encountered for $gl(m/n)$ by Jarvis and Green [12] (more precisely their adjoint identities).

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